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A homogeneous continuum without the property of Kelley

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Abstract

We generalize the property of Kelley for continua to the non-metric case. Basic properties that are true in metric case are shown to be true in general. An example is constructed showing that, unlike for metric continua, the homogeneity does not imply the property of Kelley. © 1999 Elsevier Science B.V. All rights reserved.

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1. Introduction

The property of Kelley was defined in [3] as Property 3.2 and investigated for metric continua. It was first applied to investigate hyperspace contractibility. Then the main properties of the property of Kelley were proven by R.W. Wardle in [5]. Until now it has played an important role not only in the hyperspace theory, but in the whole continuum theory. Here we extend its definition for Hausdorff continua and we verify what properties of it are valid in this wider sense. The main result is that, unlike in the metric case, the homogeneity does not imply the property of Kelley. Since the Effros property implies the property of Kelley, this is another example of a homogeneous continuum that is non-Effros, see [1].

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2. Definitions and basic properties

Given a Hausdorff continuum X we consider the hyperspace $C(X)$ of all nonempty subcontinua of X with the Vietoris topology, i.e., the topology whose basic open sets are of the form

$$\{A \in C(X): A \subset U_1 \cup \dots \cup U_n \text{ and } A \cap U_i \neq \emptyset \text{ for } i \in \{1, \dots, n\}\}$$

for any finite sequence U_1, \dots, U_n of open sets in X . We will use the notation $C^2(X)$ for $C(C(X))$ and, for any point $p \in X$, $C(p, X) = \{K \in C(X): p \in K\}$.

Given a mapping $f: X \rightarrow Y$ we define a mapping

$$C(f): C(X) \rightarrow C(Y) \quad \text{by} \quad C(f)(A) = f(A) \text{ for any } A \in C(X).$$

Note that the set A is a point of $C(X)$ in the left side of the definition, while a subset of X at the right side.

Definition 2.1. We say that a Hausdorff continuum X has the *property of Kelley at a point* $p \in X$ if for any continuum $K \in C(p, X)$ and for any open neighborhood \mathcal{U} of K in $C(X)$ there is a neighborhood U of p in X such that if $q \in U$, then there is a continuum $L \in C(X)$ with $q \in L \in \mathcal{U}$. A continuum X has the *property of Kelley* if it has the property of Kelley at each of its points.

Now we summarize the main results on the property of Kelley that are valid also for Hausdorff continua. They have been proven by R.W. Wardle in [5] for the metric case, but their proofs remain valid in the wider sense (sometimes using nets in places of sequences), or they just use definitions only, and thus are left to the reader.

A function $F: X \rightarrow C(Y)$ is said to be *upper semi-continuous at a point* $p \in X$ provided that for every open set $V \subset Y$ such that $F(p) \subset V$ there is an open set $U \subset X$ such that $p \in U$ and satisfying $F(x) \subset V$ for all $x \in U$. The function F is said to be *upper semi-continuous* if it is upper semi-continuous at each of its points.

For any continuum X define a function

$$\alpha_X: X \rightarrow C^2(X) \quad \text{by} \quad \alpha_X(p) = C(p, X).$$

Proposition 2.2 (see [5, (2.1) Theorem, p. 292]). *The function α_X is upper semi-continuous.*

Proposition 2.3 (see [5, (2.2) Theorem, p. 292]). *A continuum X has the property of Kelley (at a point $p \in X$) if and only if the function α_X is continuous (at p).*

A mapping $f: X \rightarrow Y$ between continua is called *confluent* if for any continuum Q in Y and any component C of $f^{-1}(Q)$ it is true that $f(C) = Q$. It is known that open mappings as well as monotone ones (i.e., with connected point inverses) are confluent.

Proposition 2.4 (see [5, (4.2) Theorem, p. 296]). *A function $f : X \rightarrow Y$ is confluent if and only if the diagram*

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ \alpha_X \downarrow & & \downarrow \alpha_Y \\ C^2(X) & \xrightarrow{C^2(f)} & C^2(Y) \end{array}$$

commutes.

As a consequence of Propositions 2.3 and 2.4 we have the next one.

Proposition 2.5 (see [5, (4.3) Theorem, p. 296]). *If a continuum X has the property of Kelley and a mapping $f : X \rightarrow Y$ is confluent, then Y has the property of Kelley.*

A continuum X is said to be *connected im kleinen* at a point $p \in X$ if for any neighborhood U of p there is a continuum K satisfying $p \in \text{int } K \subset K \subset U$. If a continuum is connected im kleinen at each of its points then it is locally connected.

Proposition 2.6. *If a continuum X is connected im kleinen at a point $p \in X$, then X has the property of Kelley at p .*

Proposition 2.7 (see [5, (3.1) Theorem, p. 295]). *Hereditarily indecomposable continua have the property of Kelley.*

A mapping $r : X \rightarrow Y$ of a continuum X onto its subcontinuum Y is called a *retraction* if $r|_Y$ is the identity on Y . A subcontinuum Y of a continuum X is said to be a *retract* of X if there is a retraction of X onto Y .

The following theorem has been proven in [5] for metric spaces. Here we present a non-metric proof of it.

Theorem 2.8 (see [5, (2.9) Theorem, p. 294]). *If a continuum X has the property of Kelley and Y is a retract of X , then Y has the property of Kelley.*

Proof. Let $r : X \rightarrow Y$ be a retraction and let a point $p \in Y$, a continuum $K \in C(Y)$ and an open set \mathcal{U} in $C(Y)$ satisfy $p \in K \in \mathcal{U}$ as in the definition of the property of Kelley. Then $C(r) : C(X) \rightarrow C(Y)$ is a retraction, and the set $\mathcal{V} = (C(r))^{-1}(\mathcal{U})$ is open in $C(X)$ and satisfies $p \in K \in \mathcal{V}$.

Since X has the property of Kelley, there is an open set V in X with $p \in V$ such that for any $q \in V$ there is a continuum $L \in \mathcal{V}$ with $q \in L$. Put $U = Y \cap V$ and observe that U is open in Y . Let q be any point of U and let L be a continuum as above. Then $q \in r(L) \in \mathcal{U}$, so $r(L)$ is the continuum needed in the definition of the property of Kelley for Y . This finishes the proof. \square

Now we turn our attention to homogeneous continua. We start with a definition.

Definition 2.9. A continuum X is called *homogeneous* if for any two points $p, q \in X$ there is a homeomorphism $h : X \rightarrow X$ with $h(p) = q$.

Let X be a continuum and let $H(X)$ denote the homeomorphism group of X . For any $p \in X$ define the evaluation map $E_p : H(X) \rightarrow X$ by $E_p(h) = h(p)$.

Definition 2.10. A homogeneous continuum X is said to be *Effros* if for any $p \in X$ the mapping E_p is open.

It is known that every metric homogeneous continuum is Effros (see [2,4]). In the next theorem we will prove that every Effros continuum has the property of Kelley. In [1] D.P. Bellamy and K.F. Porter have shown an example of a homogeneous continuum which is not Effros. However the continuum is locally connected, so by Proposition 2.6 it has the property of Kelley.

Theorem 2.11. *Each Effros continuum has the property of Kelley.*

Proof. Let a continuum K , a point $p \in K$ and an open set $\mathcal{U} \subset C(X)$ with $K \in \mathcal{U}$ be given as in the definition of the property of Kelley. To find an appropriate set U define $\mathcal{V} = \{h \in H(X) : h(K) \in \mathcal{U}\}$ and note that \mathcal{V} is an open neighborhood of the identity in $H(X)$. Since X is Effros, the set $U = E_p(\mathcal{V})$ is open. We will show that U satisfies the definition of the property of Kelley. In fact, if $q \in U$, then there is a homeomorphism $h \in \mathcal{V}$ satisfying $h(p) = q$. Thus $q \in h(K) \in \mathcal{U}$ and the proof is complete. \square

3. The example

In this section we will construct a homogeneous Hausdorff continuum without the property of Kelley. The construction is similar to the one in [1] by D.P. Bellamy and K.F. Porter.

Denote by $\overline{\mathbb{R}}$ the one point compactification $\mathbb{R} \cup \{\infty\}$ of the reals \mathbb{R} , and by S the unit complex circle $\{z \in \mathbb{C} : |z| = 1\}$. Thus $\overline{\mathbb{R}}$ and S are both simple closed curves, but it will simplify the description and the proofs if we distinguish them. Let us first define an auxiliary metric continuum X_1 , being a subset of the torus $\overline{\mathbb{R}} \times S$, by the formula

$$X_1 = \{(x, z) \in (\mathbb{R} \setminus \{0\}) \times S : z = \exp(2\pi i/x)\} \cup \{0\} \times S \cup \{(\infty, 1)\}.$$

The continuum X_1 is known for having the property of Kelley while its square $X_1 \times X_1$ does not have this property. In fact, the example is just the one due to S.B. Nadler, Jr. described in [5, (4.7) Example, p. 297] with the two end points identified. We can think on X_1 as on the circle $\overline{\mathbb{R}}$ with the point 0 replaced by a circle in a nice way. If we replace all points of $\overline{\mathbb{R}}$ (including ∞) by circles in the same way as we did in X_1 , we will get a homogeneous Hausdorff continuum X which can be monotonously mapped onto the continuum

X_1 . Then $X \times X$ is homogeneous again, but it does not have the property of Kelley by Proposition 2.5 since its monotone (thus confluent) image $X_1 \times X_1$ does not have it.

Precisely, the space X is defined as a torus $\overline{\mathbb{R}} \times S$ with a special topology. For any point $z_0 \in S$ and any $t > 0$ define

$$U(z_0, t) = \{z \in S: z = z_0 \exp(2\pi i s) \text{ for } s \in (-t, t)\}.$$

Thus $U(z_0, t)$ is an open neighborhood of z_0 in S with the usual topology. For $\langle x_0, z_0 \rangle \in \overline{\mathbb{R}} \times S$ and for any two positive numbers t and ε define

$$A(x_0, z_0, t) = \{x_0\} \times U(z_0, t)$$

and, for $x_0 \in \mathbb{R}$ and $\varepsilon > 0$,

$$B(x_0, z_0, t, \varepsilon) = \{\langle x, z \rangle \in \mathbb{R} \times S: x \in (x_0 - \varepsilon, x_0) \cup (x_0, x_0 + \varepsilon) \text{ and } \exp(2\pi i/(x - x_0)) \in U(z_0, t)\}.$$

Similarly,

$$B(\infty, z_0, t, \varepsilon) = \{\langle x, z \rangle \in \mathbb{R} \times S: x \in (-\infty, -\varepsilon) \cup (\varepsilon, \infty) \text{ and } \exp(2\pi i z) \in U(z_0, t)\}.$$

Then, for any $\langle x_0, z_0 \rangle \in \overline{\mathbb{R}} \times S$ and for any two positive numbers t and ε , the set

$$N(x_0, z_0, t, \varepsilon) = A(x_0, z_0, t) \cup B(x_0, z_0, t, \varepsilon)$$

is an open neighborhood of $\langle x_0, z_0 \rangle$ in X . Note that every point of $\overline{\mathbb{R}} \times S$ has a countable local basis of open neighborhoods. Therefore convergence of countable sequences suffices to determine the topology in X . The following three observations describe the convergence of sequences in X .

Observation 3.1. If $x_0 \in \overline{\mathbb{R}}$, then $\langle x_0, z_n \rangle \rightarrow \langle x_0, z_0 \rangle$ if and only if $z_n \rightarrow z_0$ in S .

Observation 3.2. If $x_0 \in \mathbb{R}$ and $x_n \neq x_0$, then $\langle x_n, z_n \rangle \rightarrow \langle x_0, z_0 \rangle$ if and only if $x_n \rightarrow x_0$ in \mathbb{R} and $\exp(2\pi i/(x_n - x_0)) \rightarrow z_0$ in S .

Observation 3.3. If $x_n \in \mathbb{R}$ for each n , then $\langle x_n, z_n \rangle \rightarrow \langle \infty, z_0 \rangle$ if and only if $x_n \rightarrow \infty$ in \mathbb{R} and $\langle 1/x_n, z_n \rangle \rightarrow \langle 0, z_0 \rangle$ in X .

4. Properties of the example

Define $p: X \rightarrow \overline{\mathbb{R}}$ by $p(\langle x, y \rangle) = x$. Then p is continuous by the definition of the topology on X , and it is monotone since $p^{-1}(x) = \{x\} \times S$.

Lemma 4.1. X is compact.

Proof. Let \mathcal{U} be any open cover of X by basic open sets. For each $x \in \overline{\mathbb{R}}$ define a finite subcollection $\mathcal{U}(x)$ of \mathcal{U} covering $\{x\} \times S = p^{-1}(x)$ in the following way. If $\{x\} \times S$

is contained in $B(y, z, t, \varepsilon)$ for some $N(y, z, t, \varepsilon) \in \mathcal{U}$, then $\mathcal{U}(x)$ is a one element set $N(y, z, t, \varepsilon)$; if there is no such set, then $\{x\} \times S$ is covered by a finite number of sets of the form $A(x_i, z_i, t_i)$ for $i \in \{1, \dots, n_x\}$ and $\mathcal{U}(x) = \{N(x_i, z_i, t_i, \varepsilon_i) \in \mathcal{U} : i \in \{1, \dots, n_x\}\}$. Then $\{\text{int } p(\bigcup \mathcal{U}(x)) : x \in \overline{\mathbb{R}}\}$ is an open cover of $\overline{\mathbb{R}}$, so it has a finite subcover $\{\text{int } p(\bigcup \mathcal{U}(x_1)), \dots, \text{int } p(\bigcup \mathcal{U}(x_n))\}$. Put $\mathcal{V} = \mathcal{U}(x_1) \cup \dots \cup \mathcal{U}(x_n)$ and observe that \mathcal{V} is a finite subcover of \mathcal{U} . The proof is finished. \square

Lemma 4.2. *X is a continuum.*

Proof. X is compact Hausdorff and $p : X \rightarrow \overline{\mathbb{R}}$ is monotone and onto, thus X is a continuum. \square

Define a function $m : X \rightarrow X_1$ by $m(\langle x, z \rangle) = \langle x, \exp(2\pi i/x) \rangle$ if $x \in \mathbb{R}$ and $x \neq 0$, $m(\langle 0, z \rangle) = \langle 0, z \rangle$ and $m(\langle \infty, z \rangle) = \langle \infty, 1 \rangle$.

Considering subsequences if necessary, we will reduce convergence of arbitrary sequences to convergence of such sequences to which Observations 3.1–3.3 can be applied.

Lemma 4.3. *The function m is continuous and monotone*

Proof. First, let $\langle x_0, z_0 \rangle \in \mathbb{R} \times S$ with $x_0 \neq 0$ and let $\langle x_n, z_n \rangle \rightarrow \langle x_0, z_0 \rangle$. If $x_n = x_0$, then $m(\langle x_n, z_n \rangle) = m(\langle x_0, z_0 \rangle)$ is a constant sequence, so it is convergent to $m(\langle x_0, z_0 \rangle)$. If $x_n \neq x_0$, then

$$m(\langle x_n, z_n \rangle) = \langle x_n, \exp(2\pi i/x_n) \rangle \rightarrow \langle x_0, \exp(2\pi i/x_0) \rangle = m(\langle x_0, z_0 \rangle).$$

Second, let $\langle x_n, z_n \rangle \rightarrow \langle \infty, z_0 \rangle$. If $x_n = \infty$, then the proof runs as before. If $x_n \neq \infty$, then $m(\langle x_n, z_n \rangle) = \langle x_n, \exp(2\pi i/x_n) \rangle \rightarrow \langle \infty, 1 \rangle$.

Third, let $\langle x_n, z_n \rangle \rightarrow \langle 0, z_0 \rangle$. If $x_n = 0$, then the proof is obvious. If $x_n \neq 0$, then, by the definition of the topology in X , we have $\exp(2\pi i/x_n) \rightarrow z_0$, so $m(\langle x_n, z_n \rangle) = \langle x_n, \exp(2\pi i/x_n) \rangle \rightarrow \langle 0, z_0 \rangle$. Thus m is continuous.

Observe that for $\langle x, z \rangle \in X_1$ the preimage $m^{-1}(\langle x, z \rangle)$ is a circle if $x \neq 0$, and it is a singleton if $x = 0$. So m is monotone. The proof is complete. \square

Corollary 4.4. *$X \times X$ does not have the property of Kelley.*

Proof. As mentioned before, the continuum $X_1 \times X_1$ does not have the property of Kelley [5, (4.7) Example, p. 297], and the map $m \times m : X \times X \rightarrow X_1 \times X_1$ is monotone (thus confluent). Now the Corollary follows from Proposition 2.5. \square

For any real number k define a function $h_k : X \rightarrow X$ in the following way. If $x \in \mathbb{R}$, then $h_k(\langle x, z \rangle) = \langle k + x, z \rangle$ and $h_k(\langle \infty, z \rangle) = \langle \infty, z \exp(2\pi i k) \rangle$.

Lemma 4.5. *For any $k \in \mathbb{R}$ the function $h_k : X \rightarrow X$ is a homeomorphism.*

Proof. It is easy to observe that h_k is one-to-one and onto. We will show its continuity. Assume $\langle x_n, z_n \rangle \rightarrow \langle x_0, z_0 \rangle$. If $x_n = x_0$ or $x_0 \in \mathbb{R}$, then the convergence $h_k(\langle x_n, z_n \rangle) \rightarrow$

$\langle x_0, z_0 \rangle$ is easy to check. If $\langle x_n, z_n \rangle \rightarrow \langle \infty, z_0 \rangle$, then, by the definition of the topology of X , we have $x_n \rightarrow \infty$ and $\exp(2\pi i x_n) \rightarrow z_0$. Thus

$$\begin{aligned} h_k(\langle x_n, z_n \rangle) &= \langle x_n + k, z_n \rangle \rightarrow \left\langle \infty, \lim_{n \rightarrow \infty} \exp(2\pi i(x_n + k)) \right\rangle \\ &= \langle \infty, z_0 \exp(2\pi i k) \rangle = h_k(\langle \infty, z_0 \rangle). \end{aligned}$$

The proof is finished. \square

Define $g : X \rightarrow X$ by $g(\langle x, z \rangle) = \langle 1/x, (\overline{z})^{x^2} \rangle$ for $x \in \mathbb{R} \setminus \{0\}$, $g(\langle 0, z \rangle) = \langle \infty, z \rangle$ and $g(\langle \infty, z \rangle) = \langle 0, z \rangle$.

Lemma 4.6. *The function g is a homeomorphism.*

Proof. It is easy to observe that g is one-to-one and onto. We will show its continuity. First, let $\langle x_0, z_n \rangle \rightarrow \langle x_0, z_0 \rangle$ for $x_0 \in \mathbb{R} \setminus \{0\}$. Then

$$g(\langle x_0, z_n \rangle) = \langle 1/x_0, (\overline{z_n})^{x_0^2} \rangle \rightarrow \langle 1/x_0, (\overline{z_0})^{x_0^2} \rangle = g(\langle x_0, z_0 \rangle).$$

Second, let $\langle x_n, z_n \rangle \rightarrow \langle x_0, z_0 \rangle$ for $x_0 \in \mathbb{R} \setminus \{0\}$ and $x_n \neq x_0$. Then

$$\exp(2\pi i(x_n - x_0)) \rightarrow z_0$$

and we have

$$\begin{aligned} g(\langle x_n, z_n \rangle) &= \langle 1/x_n, (\overline{z_n})^{x_n^2} \rangle \rightarrow \left\langle 1/x_0, \lim_{n \rightarrow \infty} \exp(2\pi i(1/x_n - 1/x_0)) \right\rangle \\ &= \left\langle 1/x_0, \lim_{n \rightarrow \infty} \exp(2\pi i x_0 x_n / (x_0 - x_n)) \right\rangle = \langle 1/x_0, (\overline{z_0})^{x_0^2} \rangle = g(\langle x_0, z_0 \rangle). \end{aligned}$$

Third, let $\langle x_n, z_n \rangle \rightarrow \langle 0, z_0 \rangle$ for $x_n \neq 0$. Then $\lim_{n \rightarrow \infty} \exp(2\pi i/x_n) = z_0$, and thus

$$g(\langle x_n, z_n \rangle) = \langle 1/x_n, (\overline{z_n})^{x_n^2} \rangle \rightarrow \left\langle \infty, \lim_{n \rightarrow \infty} \exp(2\pi i/x_n) \right\rangle = \langle \infty, z_0 \rangle = g(\langle 0, z_0 \rangle).$$

Fourth, let $\langle 0, z_n \rangle \rightarrow \langle 0, z_0 \rangle$. Then $z_n \rightarrow z_0$, and we have

$$g(\langle 0, z_n \rangle) = \langle \infty, z_n \rangle \rightarrow \langle \infty, z_0 \rangle = g(\langle 0, z_0 \rangle).$$

Fifth, let $\langle \infty, z_n \rangle \rightarrow \langle \infty, z_0 \rangle$. Then $z_n \rightarrow z_0$, and we have

$$g(\langle \infty, z_n \rangle) = \langle 0, z_n \rangle \rightarrow \langle 0, z_0 \rangle = g(\langle \infty, z_0 \rangle).$$

Sixth, and the last, let $\langle x_n, z_n \rangle \rightarrow \langle \infty, z_0 \rangle$ and $x_n \neq \infty$. Then

$$\lim_{n \rightarrow \infty} \exp(2\pi i x_n) = z_0$$

and we have

$$g(\langle x_n, z_n \rangle) = \langle 1/x_n, (\overline{z_n})^{x_n^2} \rangle \rightarrow \left\langle 0, \lim_{n \rightarrow \infty} \exp(2\pi i x_n) \right\rangle = \langle 0, z_0 \rangle = g(\langle \infty, z_0 \rangle).$$

The proof is finished. \square

Corollary 4.7. *The continuum X is homogeneous.*

Proof. It is enough to show a homeomorphism moving the point $\langle 0, 0 \rangle$ to any other point $\langle x_0, z_0 \rangle \in X$. First, assume that $x_0 \neq \infty$. Then the following sequence of homeomorphisms moves $\langle 0, 1 \rangle$ to $\langle x_0, z_0 \rangle$:

$$\langle 0, 1 \rangle \xrightarrow{g} \langle \infty, 1 \rangle \xrightarrow{h_{\text{Arg } z_0}} \langle \infty, z_0 \rangle \xrightarrow{g} \langle 0, z_0 \rangle \xrightarrow{h_{x_0}} \langle x_0, z_0 \rangle.$$

If $x_0 = \infty$, then even shorter sequence of homeomorphisms is enough. \square

Summarizing results of this section we have the following main result.

Theorem 4.8. *The continuum $X \times X$ is homogeneous without having the property of Kelley.*

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